Math 10B - Calculus of Several Variables II - Spring 2011
May 27, 2011
Practice Final

Name: Solutions

There are 160 possible points on this exam. There is no need to use calculators on this exam. All electronic devices should be turned off and put away. The only things you are allowed to have are: a writing utensil(s) (pencil preferred), an eraser, and an exam. All answers should be given as exact, closed form numbers as opposed to decimal approximations (e.g., π as opposed to 3.14159265358979...). Cheating is strictly forbidden. You may leave when you are done. Good luck!

Problem	Score
1	/10
2	/10
3	/30
4	/30
5	/20
6	/20
7	/20
8	/10
9	/10
Score	/160

Problem 1 (10 points). *The average value of a function $f(x,y)$ over a region R is the value*

$$AV(f,R) = \frac{1}{Area(R)} \iint_R f(x,y)\, dA.$$

Find the average value of $f(x,y) = x \sin xy$ over the region $R = [0,1] \times [0,1]$.

Area $(R) = 1$ (area of a square)

$$AV(f,R) = \frac{1}{1} \iint_R x \sin xy \, dA = \int_0^1 \int_0^1 x \sin xy \, dy\, dx$$

$$= \int_0^1 (-\cos xy)\Big|_0^1 \, dy = \int_0^1 (1 - \cos y)\, dy$$

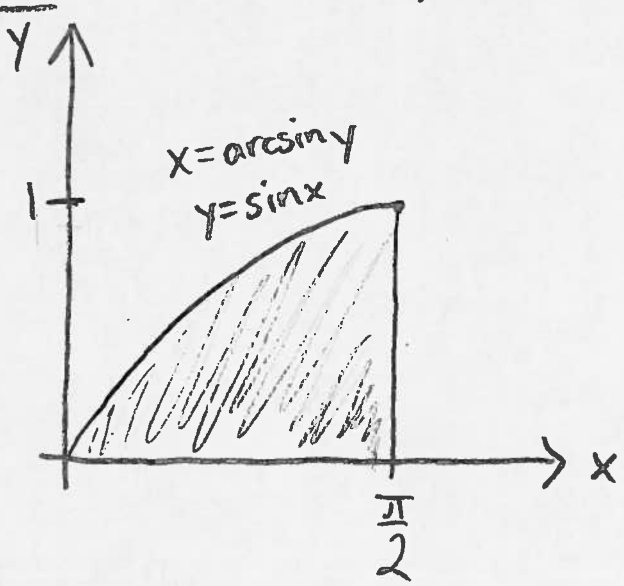
$$= (y - \sin y)\Big|_0^1 = \boxed{1 - \sin 1}$$

Problem 2 (10 points). *Compute the integral*

$$\int_0^1 \int_{\arcsin(y)}^{\frac{\pi}{2}} e^{\cos x} dx dy$$

and draw the region of integration.

Sketch : $x = \arcsin y \iff y = \sin x \quad \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right)$



$$\int_0^1 \int_{\arcsin(y)}^{\frac{\pi}{2}} e^{\cos x} dx dy = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} e^{\cos x} dy dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin x) e^{\cos x} dx \qquad \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array}$$

$$= \int_1^0 -e^u du = -e^u \Big|_1^0 = (-1) - (-e) = \boxed{e - 1}$$

Problem 3 (30 points).

(a) *(10 points) Show that* $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$ *where* (r,θ) *is polar coordinates.*

(b) *(10 points) Combine*

$$\int_0^2 \int_0^x \sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$$

1 2

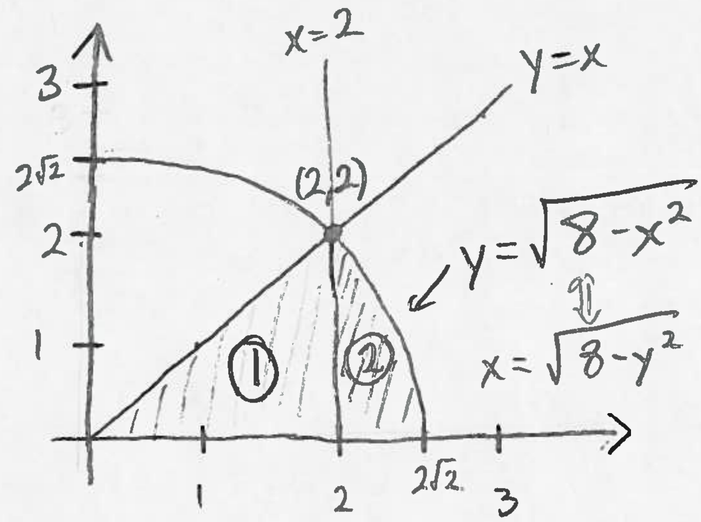
(Hint: Use polar coordinates.)

(c) *(10 points) Compute the integral in part (b).*

a $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

b Sketch:



@ $x=2$

$y = \sqrt{8-x^2}$

$= \sqrt{8-4} = \sqrt{4} = 2$

$$\int_0^2\int_0^x \sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \boxed{\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} r(r\,dr\,d\theta)}$$

OR

(could also switch order: $\boxed{= \int_0^2\int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\,dx\,dy}$)

c Need polar to compute

$$\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{3}r^3\Big|_0^{2\sqrt{2}} d\theta = \int_0^{\frac{\pi}{4}} \frac{16\sqrt{2}}{3}\,d\theta = \boxed{\frac{4\sqrt{2}\,\pi}{3}}$$

Problem 4 (30 points).

(a) *(10 points) State the equations to convert Cartesian coordinates to cylindrical coordinates and give the Jacobian of this transformation.*

(b) *(10 points) State the equations to convert Cartesian coordinates to spherical coordinates and give the Jacobian of this transformation.*

(c) *(10 points) Show that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}dxdydz = 2\pi.$$

(a) $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

(b) $x = \rho\cos\theta\sin\varphi$, $y = \rho\sin\theta\sin\varphi$, $z = \rho\cos\varphi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2\sin\varphi$$

(c)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\; e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \rho e^{-\rho^2}(\rho^2\sin\varphi)\,d\varphi\,d\theta\,d\rho$$

$$= \int_0^{\infty}\int_0^{2\pi} -\rho^3 e^{-\rho^2}\cos\varphi\Big|_0^{\pi}\,d\theta\,d\rho = \int_0^{\infty}\int_0^{2\pi} 2\rho^3 e^{-\rho^2}\,d\theta\,d\rho$$

$$= 4\pi\int_0^{\infty}\rho^3 e^{-\rho^2}\,d\rho \qquad \begin{pmatrix} u=\rho^2 & dv = \rho e^{-\rho^2}d\rho \\ du = 2\rho d\rho, & v = -\frac{1}{2}e^{-\rho^2}\end{pmatrix}$$

$$\lim_{t\to\infty}\left(4\pi\left[-\frac{1}{2}\rho^2 e^{-\rho^2}\right]_0^t + \int_0^t \rho e^{-\rho^2}d\rho\right]\Bigg) = \lim_{t\to\infty}\left(4\pi\left[-\frac{1}{2}\rho^2 e^{-\rho^2} - \frac{1}{2}e^{-\rho^2}\right]\Big|_0^t\right)$$

$$= \lim_{t\to\infty}\left(-2\pi\left[\left(\cancel{t^2e^{-t^2}}^{\to 0} + \cancel{e^{-t^2}}^{\to 0}\right) - (0+1)\right]\right) = -2\pi(-1) = \boxed{2\pi}$$

Problem 5 (20 points).

(a) *(10 points) Determine whether* $\vec{F}(x,y) = (2xy, x^2)$ *is conservative. If so, find a potential for* $\vec{F}$.

(b) *(10 points) Compute* $\int_c \vec{F}\cdot d\vec{s}$ *where* c *is the portion of* $y = x^{17} + 2x^{15} - 2x^{10} + 7x^7 - 3x^5 + \pi$ *from* $(0,\pi)$ *to* $(1, 5+\pi)$.

ⓐ $\vec{F} = \langle P, Q\rangle$ if $\vec{F}$ is conservative, $\vec{F} = \nabla f$

$f = \int P\,dx = x^2y + g(y)$, $f_y = x^2 + g'(y) = Q = x^2$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c$. Choose $c = 0$.

A potential is $\boxed{f = x^2y}$

ⓑ $\vec{F} = \nabla f$, $f = x^2y$

Fundamental theorem of line integrals gives:

$$\int_c \vec{F}\cdot d\vec{s} = f(1, 5+\pi) - f(0,\pi) = (1)^2(5+\pi) - (0)^2(\pi) = \boxed{5+\pi}$$

Problem 6 (20 points).

(a) *(10 points) State Green's theorem.*

(b) *(10 points) If f and g are differentiable functions and* $\mathbf{c}$ *is a piecewise smooth, simple, closed path, show:*

$$\int_{\mathbf{c}} f(x)\,dx + g(y)\,dy = 0.$$

ⓐ Suppose C is a piecewise smooth, closed, simple, positively oriented curve which bounds a region D. If $P(x,y)$ & $Q(x,y)$ are C' functions on an open set containing D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

ⓑ $\frac{\partial f}{\partial y} = 0$, $\frac{\partial g}{\partial x} = 0$, So by Green's thm:

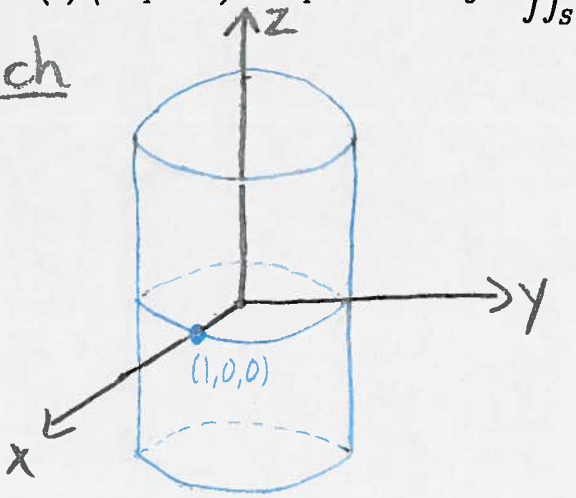
$$\int_C f(x)\,dx + g(y)\,dy = \iint_D 0\,dA = 0.$$

Problem 7 (20 points). *Consider the surface $x^2 + y^2 = 1$, $-1 \le z \le 1$. Call this surface S and orient it with normals that point away from the axis of the cylinder.*

(a) *(10 points) Find the tangent plane at* $(1,0,0)$.

(b) *(10 points) Compute the integral* $\iint_S \vec{F} \cdot d\vec{S}$ *where* $\vec{F}(x,y,z) = (x,y,0)$.

Sketch



ⓐ Parametrize first: $\vec{r}(\theta,z) = \langle \cos\theta, \sin\theta, z\rangle$, $0 \le \theta \le 2\pi$, $-1 \le z \le 1$

$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0\rangle$, $\vec{r}_z = \langle 0,0,1\rangle$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0\rangle$$

$\vec{r}(0,0) = \langle 1,0,0\rangle$ so a normal vector to the tangent plane at $(1,0,0)$ is $\vec{r}_\theta \times \vec{r}_z(0,0) = \langle 1,0,0\rangle$.

Tangent plane: $\langle 1,0,0\rangle \cdot \langle x-1, y, z\rangle = x-1 = 0 \iff \boxed{x=1}$

ⓑ In part ⓐ we see $\vec{r}_\theta \times \vec{r}_z$ points in the right direction, so

$$\iint_S \vec{F} \cdot d\vec{S} = \int_{-1}^{1}\int_0^{2\pi} \langle \cos\theta, \sin\theta, 0\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle \, d\theta\, dz$$

$$= \int_{-1}^{1}\int_0^{2\pi} d\theta\, dz = \boxed{4\pi}$$

Problem 8 (10 points). *Let S be the sphere of radius 2, centered at (a, b, c). Orient S with outward normals. Let $\vec{F}(x, y, z) = (x, y, z)$. Compute the integral*

$$\iint_S \vec{F} \cdot d\vec{S}.$$

Since S is positively oriented, by the divergence theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_B (\operatorname{div} \vec{F})\, dV$$

where $\partial B = S$.

$\operatorname{div} \vec{F} = 1+1+1 = 3$, so

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_B 3\, dV = 3 \cdot \operatorname{Vol}(B) = \cancel{3} \cdot \left(\frac{4}{\cancel{3}} \pi (2)^3\right) = \boxed{32\pi}$$

$(-y, x, z)$

Problem 9 (10 points). *Compute* $\iint_S \mathrm{curl}\,\vec{F}\cdot d\vec{S}$ *where* $\vec{F}(x,y,z) =$ [redacted] *and* S *is the surface* $x^2+y^2+z^2 = 9^2$, $z \geq 0$, *with upward normals.*

By Stokes' theorem, $\iint_S \mathrm{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$

where C has the induced orientation from S.

In this case, C is the circle $x^2+y^2=9^2$, $z=0$ oriented counterclockwise. C is parametrized by

$\vec{r}(t) = \langle 9\cos t, 9\sin t, 0\rangle$, $0 \leq t \leq 2\pi$.

$\vec{F}(\vec{r}(t)) = \langle -9\sin t, 9\cos t, 0\rangle$, $\vec{r}\,'(t) = \langle -9\sin t, 9\cos t, 0\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 81\sin^2 t + 81\cos^2 t + 0 = 81$

$$\iint_S \mathrm{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 81\,dt = \boxed{162\pi}$$